Detecting and localizing multipartite entanglement by two-partite methods

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Abstract: A notion of partial entanglement in multipartite systems is discussed. Some algorithms for detecting and localizing the introduced partial entanglement in general multipartite quantum states are presented. The main feature of the methods presented is that they are exclusively based on two-partite methods.

Keywords: entanglement of quantum states, partial entanglement, bipartite methods for entanglement detection

1. INTRODUCTION

Quantum entanglement, since its discovery by Einstein, Podolsky and Rosen [1], and Schrodinger [2] attracted very large attention, especially in the past two decades due to its intriguing properties and extremely important significance in the quantum information processing tasks [3, 4, 11]. The main efforts have been focused on the bipartite entanglement and at present many interesting and deep results have been obtained in this case, see [5, 6] for some recent reviews.

The present note is addressed to a very preliminary discussion of certain aspects of multipartite entanglement. The importance of multipartite entanglement can be illustrated by several illuminating examples. A class of the so called graph states used in one-way quantum computation model [7] and fault-tolerant topological quantum computation model as well are both good examples. Multipartite entangled photons are essential resources used in quantum key distribution protocols [8,9]. For the purposes of quantum communication a multipartite entangled states can serve as quantum communication channels in most of the known teleportation protocols [10]. Multipartite entanglement displays one of the most fascinating features of quantum theory that is called as nonlocality of the quantum world [13].

Comparing to the best known two-partite case [4,5,6], the situation concerning a general multipartite system is much more complicated even on the level of classification of the possible entanglements that can arise in such systems. Given a multipartite state, a natural question to ask is whether it is entangled state and if so how to measure the amount of entanglement included.

The main aim of the present note is to demonstrate how one can use exclusively two-partite systems methods to answer the first question for a general multipartite quantum state.

The present note is organized in the following way. In next section basic definitions will be introduced and the algorithm detecting and localizing entanglements in the case of pure states will be formulated. An extension of this method to general multipartite mixed states will be presented in Section (3).

2. A GENERAL MULTIPARTITE SYSTEMS. THE CASE OF PURE STATES

Let Q be a quantum system composed from n smaller and separated subsystems. The corresponding Hilbert spaces H_i of the subsystems are assumed to be finite dimensional with the corresponding dimensions k_i for i=1,…n.

Then the resulting Hilbert space H describing possible states of the total system is equal to $H = H_1 \otimes H_2 \otimes \ldots \otimes H_n$ and has dimension $K = k_1 \cdot k_2 \cdot \ldots \cdot k_n$. The space of all quantum states i.e. the space of all nonnegative ,of trace equal to one endomorphisms of the corresponding Hilbert space of states H will be denoted by $E(H)$ and its boundary $\partial E(H)$ is composed from pure states of the system Q.

Let $\wp(n)$ stands for the set of all partitions of the set $I_n = \{1,2,\ldots,n\}$. For a particular $\pi = (X_1,\ldots,X_k) \in \wp(n)$ we denote the number of elements of π, $|\pi| = k$. A partial semiorder $\prec_f$ in $\wp(n)$ is introduced by the following definition: we say that $\pi = (X_1, X_2, \ldots, X_k)$ is finer than a partition $\pi' = (X_1', X_2', \ldots, X_l')$, denoted as $\pi \prec_f \pi'$ iff for any $i \in \{1,\ldots,k\}$ there exists $j \in \{1,\ldots,l\}$ such that $X_i \subseteq X_j'$. The maximal element $\pi_{\max}$ exists in the poset $(\wp(n), \prec_f)$ and $\pi_{\max} = (I_n)$ and the corresponding minimal partitions $\pi_{\min} = (\{1\},\{2\},\ldots,\{n\})$.

The cardinality of the set $\wp(n)$ denoted as $|\wp(n)|$ is given by the so called Bell number $B(n)$. The number of all partitions of the set I_n of length k is given by Stirling number of second kind $S(n,k)$. In particular $S(n,2) = 2^{n-1} - 1$ and $B(n) = \sum_{k=1}^{n} S(n,k)$.

With these preparatory definitions we can pass to the basic notions.

A pure state $|\psi\rangle \in \partial E(H_1 \otimes H_2 \otimes \ldots \otimes H_n)$, $n \geq 2$ is called π-separable iff

- $|\psi\rangle = \bigotimes_{i=1}^{k} |\psi_i\rangle$ for $\pi = (X_1, X_2, \dots, X_k) \in \wp(n)$, $|\psi_i\rangle \in H(X_i)$ and where $H(X_i)$ stands for the Hilbert space corresponding to the X_i-piece of the whole system,
- there is no finer partition π' for which $|\psi\rangle$ is π'-separable.

Remark:

There is some abuse of notation introduced in the above definition which however can be easily overcomed when passing to the corresponding matrix representation of $|\psi\rangle$.

In particular a state $|\psi\rangle$ is called completely separable iff $|\psi\rangle$ is $\pi_{\min}$-separable. Any state $|\psi\rangle$ which is not $\pi_{\min}$-separable is partially entangled state as the following definition state.

Definition:

A state $|\psi\rangle \in \partial E(H_1 \otimes H_2 \otimes \ldots \otimes H_n)$ is called π-entangled state iff there exists $\pi \in \wp(n)$, $\pi \succ_f \pi_{\min}$ such that $|\psi\rangle$ is π-separable. In particular a state $|\psi\rangle$ which is $\pi_{\max}$-separable will be called completely entangled state.

The first question when dealing with a particular state $|\psi\rangle$ is the question whether this state is separable or entangled state.

In the case of two-partite systems and for vector states the complete answer to this question is provided by the corresponding Schmidt decomposition [3, 4]. However in general n-partite case there is no canonical notion of Schmidt decomposition as is well known.

The algorithm formulated below gives a systematic way to answer this question in general and additionally localizes the finest partition for which the separability holds.

To start with, we introduce the following construction, which can be easily implemented in any computer system which contains any of the standard Linear Algebra Package.

The canonical Schmidt decomposition:

Let H_1, H_2 be two Hilbert spaces of dimension n_1, resp. n_2. Then for any $|\psi\rangle \in H = H_1 \otimes H_2$, there exists an unique decomposition:

$$|\psi\rangle = \sum_{i=1}^{r(|\psi\rangle)} \lambda_i |\psi_i^1\rangle \otimes |\psi_i^2\rangle \qquad (1)$$

where the number $1 \le r(|\psi\rangle) \le \min(n_1, n_2)$ is the Schmidt rank of $|\psi\rangle$ and $\{|\psi\rangle_i^\alpha, i = 1, 2, \ldots n_\alpha\}$ form complete orthonormal systems (CONS) in the corresponding spaces.

Function SD

Input:

$n, n_1, n_2, |\psi\rangle$ where $n = n_1 n_2, |\psi\rangle \in C^{n_1 n_2}$

Output:

$[r, (|\psi_i^1\rangle)_{i=1,\ldots,n_1}, (|\psi_i^2\rangle)_{i=1,\ldots,n_2}]$

$[r, (|\psi_i^1\rangle)_{i=1,\ldots,n_1}, (|\psi_i^2\rangle)_{i=1,\ldots,n_2}] = SD(|\psi\rangle, (n_1, n_2))$

We will also use the following function which computes the complete list of all 2-partitions of a given set $X = \{x_1, x_2, \ldots, x_n\}$.

Let us mention that it is not quite trivial to formulate an algorithm which computes the list of desirable two-partitions of a given n-th element set $X = \{x_1, x_2, \ldots, x_n\}$. However using the methods presented in [12] this goal can be obtained.

Function 2p-Par

Input:

$X = \{x_1, x_2, \ldots, x_n\}$

Output: $2\wp p$, the list of all 2-partions of X

$[(X_i, Z_i), i = 1, 2, \ldots)] = 2p - Par(X)$

Now we are ready to formulate heuristic version of our algorithm.

Algorithm: Separable of Entangled?
The case of vector states

Input:

$H = H_1 \otimes H_2 \otimes \ldots \otimes H_n$,

$n = k_1 \cdot k_2 \cdot \ldots \cdot k_n, |\psi\rangle \in \partial E(H_1 \otimes H_2 \otimes \ldots \otimes H_n)$

where $k_i = \dim H_i$, $H_i \cong C^{k_i}$

Output:

$\pi \in \wp(n)$ such that $|\psi\rangle$ is π-separable, in particular if $\pi = \pi_{\min}$ then $|\psi\rangle$ is completely separable.

Algorithm:

Set $X = I_n = \{1, 2, \ldots n\}; \pi = [];$ % $\pi = \varnothing$;
while $|X| \ge 1$
  set $N = |X|$
  for k=1:N,
    if $|X[k]| = 1$ then
      $X = X \setminus X[k]$
      $\pi = [\pi, X[k]]$
    end if
  end for
  set $N = |X|$
  for k=1:N,
    $2pL[k] \leftarrow 2p - Par\, X[k]$ *%form a list of all 2-partitions of* $X[k]$
  end for
  set $l = 1$
  while $l < |2pL[k]|$
    for $(Y, Z) \in 2pL[k]$
      $(r_{Y,Z}, |\psi_Y, |\psi_Z\rangle\rangle) \leftarrow SD(|\psi_{X[k]}\rangle, (Y, Z))$
      If $r_{Y,Z} = 1$ then
        set $X[k] = (Y, Z)$
        stop for
      else
        go to the next $(Y, Z) \in 2pL[k]$
        $l = l + 1$

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        end if
      end for
    end while
    if l = |2pL[k]| then
        X = X \ X[k]
        π = [π, X[k]]
    end if
    end for
end while
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3. A GENERAL MULTIPARTITE SYSTEM. THE CASE OF MIXED STATES

A general state $\rho \in E(H_1 \otimes H_2 \otimes \ldots \otimes H_n)$ will be called π-separable state, for some $\pi = (X_\alpha, \alpha = 1, \ldots k) \in \wp(n)$, iff there exists a representation of ρ of the form

$$\rho = \sum_\alpha P_\alpha \bigotimes_{X \in \pi} \rho_\alpha^X, \qquad (2)$$

where $\rho_\alpha^X \in E(H^X)$ for each α and moreover ρ is not π'-separable for any π' finer that π.

There exists plenty of different criterions answering (mostly partially only) the question whether a given state of bipartite system is separable or entangled. Without specifying the concrete form of the criterion used we assume that an appropriate tool for this purpose is available. It will be named as 2p-Oracle.

2p-Oracle function

Input:
$H = H_A \otimes H_B$ where $\rho \in E(H_A \otimes H_B)$

Output:
YES if ρ is entangled, NO if ρ is separable

Remark:

The best known examples of such oracles although working only for a particular classes of states are: PPT-criterion, certain witness construction, see [5, 6, 4]. In general the problem of answering the question whether a given quantum state ρ is entangled or not is NP-Hard problem [17].

Now, we are ready to formulate our algorithm by which we can answer the question whether a given general state $\rho \in E(H_1 \otimes H_2 \otimes \ldots \otimes H_n)$ is completely or partially separable and in the second case the localization of the corresponding $\pi \in \wp(n)$ is obtained as well.

Algorithm: Separable of Entangled? General n-partite states

Input:
$H = H_1 \otimes H_2 \otimes \ldots \otimes H_n$,
$n = k_1 \cdot k_2 \cdot \ldots \cdot k_n$, $\rho \in E(H_1 \otimes H_2 \otimes \ldots \otimes H_n)$ where
$k_i = \dim H_i$, $H_i \cong C^{k_i}$

Output:
$\pi \in \wp(n)$ such that ρ is π-separable

Algorithm:

```
Set X = I_n = {1,2,...,n}; π = [];   % π = ∅
while |X| ≥ 1
  set N = |X|
  for k=1:N,
    if |X[k]| = 1 then
        X = X \ X[k]
        π = [π, X[k]]
    end if
  end for
  set N = |X|
  for k=1:N,
    2pL[k] ← 2p − Par X[k]
    set l = 1
    while l < |2pL[k]|
      for (Y,Z) ∈ 2pL[k]
        (r_{Y,Z}, |ψ_Y, |ψ_Z⟩⟩) ← SD(|ψ_{X[k]}⟩, (Y,Z))
        If 2p − Oracle(ρ,Y,Z) = NO then
          set X[k] = (Y,Z)
          stop for
        else
          go to the next (Y,Z) ∈ 2pL[k]
          l = l + 1
        end if
      end for
    end while
    if l = |2pL[k]| then
      X = X \ X[k]
      π = [π, X[k]]
    end if
  end for
end while
```

4. SUMMARY

A general method for detecting and localizing partial and complete as well entanglement for general multipartite states is presented.

It was shown that the question of detection and localizing (partial) entanglement of quantum states describing multipartite systems can be answered by using exclusively tools worked out for two-partite case. However the serious drawback of methods introduced here is the non-polynomial computational complexity of our algorithms presented. A rough estimate gives quickly the computational complexity of both algorithms presented as $O(2^n)$. The source of such big complexity is the length of the list of all two-partitions of a given set X.

However, for small quantum registers composed of small number n ~ 20 – 25 quantum logical units like qubits or qudits which is the case of all quantum computer simulators available at present (see [14]) our methods might be quite useful.

The methods introduced in the present contribution could find straightforward applications to several problems of quantum information processing tasks. Let us mention two of them:

- in the process of tracing of entanglement when executing quantum programs on Quantum Computer Simulators,
- in the search for a new effectively simmulable quantum circuits by using the methods of [15].

A natural semiorder relation with respect to the amount of entanglement included is that induced by the action of (S)LOCC-class of operations. In the case of two-partite systems this notion seems to be quite well understood [4,5,6] although several basic question are still to be resolved even in this case. However, in the case of multipartite systems several problems arise at the very beginning. For example, let us consider two quantum states ρ_1, ρ_2 of a composite nth-partite system and let us assume that ρ_1 is π_1-entangled and ρ_2 is π_2-entangled where of course $\pi_i = (X_1^i, X_2^i, \ldots, X_k^i), i = 1,2$ are the corresponding partitions of the set of indices $\{1,2,3,\ldots,n\}$.

From the very definitions of (S)LOCC class of operation we must restrict ourselves to the situation $\pi_1 \prec_f \pi_2$ or $\pi_2 \prec_f \pi_1$ in order to define the local actions on both states ρ_1 and ρ_2. So, let us assume that $\pi_1 = (X_1, X_2, \ldots, X_k) \prec \pi_2 = (Y_1, Y_2, \ldots, Y_l)$. By (S)LOCC-class of operations we mean local with respect to the finer decomposition $(X_1, X_2, \ldots, X_k)$ completely positive operators together with general but local measurement equipments (supported with classical communications channels) [18]. We say that $\rho_1 \underset{(S)LOCC}{\prec} \rho_2$ iff there exists (S)LOCC-class operations that transforms ρ_2 into the state ρ_1. From the above definitions it follows that we should expect much linear chains in the $POSET(E(H_1 \otimes H_n), \underset{(S)LOCC}{\prec})$ then in the bipartite case. In particular it follows from [16] that there exists many locally maximal states with respect to this semiorder.

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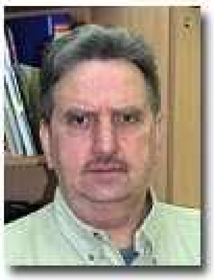

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