

Stability of time-varying linear system

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Abstract: In this paper we consider sufficient conditions for the exponential stability of linear time-varying systems with continuous and discrete time. Stability guaranteeing upper bounds for different measures of parameter variations are derived.

Keywords: time-varying linear systems, exponential stability, stability of linear systems, asymptotic stability, Lapunov exponent.

Streszczenie: W artykule rozważane są warunki wystarczające na eksponencjalną stabilność dla zmiennych w czasie układów liniowych zarówno ciągłych jak i dyskretnych. Prezentowane są różne miary zapewniające stabilność oraz wyprowadzone zostało górne ograniczenie na zmienność parametrów zapewniające stabilność. (Stabilność zmiennych w czasie układów liniowych)

Słowa kluczowe: zmienne w czasie układy liniowe, eksponencjalna stabilność, stabilność układów liniowych, stabilność asymptotyczna, wykładnik Lapunowa.

1. INTRODUCTION

Stability and stability conditions are one of the most important problem in a system projecting process. Ensuring the stability for systems is a main issue that decides the correct action. During the construction, subsequent tests and simulations, it is necessary to check whether the test object is stable and/or we need to define the settings for which this system is stable. In determining stability, useful and quick to assess are the stability conditions. For stationary systems stability conditions are well known. There are many methods described in the literature for such systems that determine conditions for the system stability. For complex or nonstationary systems where state depends on switching signal the stability conditions are not so obvious and problem of ensuring stability or determining whether the system is stable or not, is not an easy and trivial task.

In this paper we study problems concerning exponential stability of linear time-varying system of the form:

$$\dot{x}(t) = A(t)x(t), t \geq 0 \quad (1)$$

and

$$x(k+1) = A(k)x(k), n \geq 0 \quad (2)$$

If the function $A(t)$ in (1) is piecewise constant then system (1) is called switched linear system. If all of the matrices $A(t)$ are Hurwitz, then it is possible to ensure the stability of the associated switched system by switching sufficiently slowly between the asymptotically stable

constituent appropriate time invariant systems. This means that instability arises in (1) as a result of rapid switching between these vector fields. Given this basic fact, a natural and obvious method to ensure the stability of (1) is to somehow constrain the rate at which switching takes place. The basic idea of constraining the switching rate has appeared in many studies on time varying systems over the past number of decades [6, 8, 13]. One of the best known and most informative of these studies was given by Charles Desoer in 1969 in his study of slowly varying systems [4]. The basic problem considered by Desoer was to find conditions on the switching rate that would ensure the stability of an unforced system of the form (1), where $A(t)$ is a matrix valued continuous function such that for some $a > 0$ the condition $\text{Re } \sigma(A(t)) < -a$ is satisfied. There are two key points to emphasize here; firstly, the stability of the time-varying system can be ensured by suitably constraining the rate of variation of $A(t)$, and secondly the constraint on derivative of $A(t)$ is determined by a Lyapunov function associated with the system.

It is well-known that if, for each $t \geq 0$, $(k \geq 0)$, all eigenvalues of $A(t)$, $(A(k))$ are lying in the proper open left half complex plane (in open unite circle), then the system (1), ((2)) is not necessarily exponentially stable (see e.g. [12]). Exponential stability is secured if, additionally, the parameter variation of, $A(t)$, $(A(k))$ is "slow enough", see [2, 11]. However, these are qualitative results. In [8] quantitative results are derived for continuous time. This means, upper bounds for the eigenvalues and for the rate of change of $A(t)$ which ensure exponentials stability of (1) are determined. Results of this kind are presented in Section 2. It seems that similar results for discrete time are unknown. In this paper we present such results.

In Section 2. previous achievements and deriver stability conditions in continuous time systems are presented, important theorem was in new, differently way proved, there are also examples of continuous time systems shown. In Section 3. similar conditions for discrete time systems are derived and illustrated with numerical example in Section 4. Section 5. includes conclusion.

2. CONTINUOUS TIME SYSTEM

Consider the homogeneous linear time-varying differential equation (1), with $A(t) \in PC(R_+, R^{n \times n})$, where $PC(R_+, R^{n \times n})$ denotes the set of piecewise continuous real n by n matrix functions on $R_+ = [0, \infty)$. Let $\langle \bullet, \bullet \rangle$ be the usual inner product on R^n , $\|\bullet\|$ the associated norm

and $\|B\|$ induced operator norm for any linear operator $B \in L(R^n, R^n)$. Denote by $\Phi(t, s)$ the transition matrix of (1) and by $X(t) = \Phi(t, 0)$ fundamental matrix. We will assume that $A(t)$ is bounded i.e there is a constant M such that

$$\|A(t)\| < M \text{ for all } t \geq 0. \quad (3)$$

Definition 1 The system (1) is said to be exponentially stable if there exist $C, \omega > 0$ such that

$$\|\Phi(t, s)\| \leq Ce^{-\omega(t-s)} \text{ for all } t \geq s \geq 0. \quad (4)$$

It appears that exponential stability can be characterized in terms of Bohl exponents.

Definition 2 The Bohl exponent $\beta(A)$ of the system (1) is given by

$$\beta(A) = \limsup_{s, t \rightarrow \infty} \frac{\ln \|\Phi(t, s)\|}{t - s}. \quad (5)$$

It can be shown [3] that for the Bohl exponent we have the following formula

$$\beta(A) = \inf \left\{ -\omega : \exists M_\omega > 0 \forall t > s > 0 \quad \|\Phi(t, s)\| \leq M_\omega e^{-\omega(t-s)} \right\}. \quad (6)$$

Moreover the Bohl exponent is upper semi-continuous what is crucial in most perturbation questions and system (1) is exponentially stable if and only if $\beta(A) < 0$.

If $A(t)$ is a constant matrix it is well-known that (1) is exponentially stable iff the real parts of the eigenvalues of A are lying in the open left half plane. For time-varying systems, even if they are analytic and periodic, exponential stability does neither imply

$$\operatorname{Re} \sigma(A(t)) \subset C^- \quad (7)$$

nor does for some $a > 0$ the condition

$$\operatorname{Re} \sigma(A(t)) < -a \text{ for all } t \geq 0 \quad (8)$$

guarantee exponential stability.

Example 1 Let [7]

$$A(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} -1 & -5 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}. \quad (9)$$

Then $\sigma(A(t)) = \{-1\}$ for all $t \geq 0$ and it can be easily verified that a fundamental matrix is given by

$$X(t) = \begin{bmatrix} e^t \left(\cos t + \frac{1}{2} \sin t \right) & e^{-3t} \left(\cos t - \frac{1}{2} \sin t \right) \\ e^t \left(\cos t - \frac{1}{2} \sin t \right) & e^{-3t} \left(\cos t + \frac{1}{2} \sin t \right) \end{bmatrix}. \quad (10)$$

Thus (1) is not exponentially stable.

Example 2 Let [12]

$$A(t) = \begin{bmatrix} -\frac{11}{2} + \frac{15}{2} \sin 12t & \frac{15}{2} \cos 12t \\ \frac{15}{2} \cos 12t & -\frac{11}{2} - \frac{15}{2} \sin 12t \end{bmatrix}. \quad (11)$$

Then $\sigma(A(t)) = \{2, -13\}$ for all $t \geq 0$ and it can be easily verified that a fundamental matrix is given by

$$X(t) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (12)$$

where

$$a_{11} = \frac{1}{2} e^{-t} (\cos 6t + 3 \sin 6t) + \frac{1}{2} e^{-10t} (\cos 6t - 3 \sin 6t)$$

$$a_{12} = \frac{1}{6} e^{-t} (\cos 6t + 3 \sin 6t) - \frac{1}{6} e^{-10t} (\cos 6t - 3 \sin 6t)$$

$$a_{21} = \frac{1}{2} e^{-t} (3 \cos 6t - \sin 6t) - \frac{1}{2} e^{-10t} (3 \cos 6t + \sin 6t)$$

$$a_{22} = \frac{1}{6} e^{-t} (3 \cos 6t - \sin 6t) + \frac{1}{6} e^{-10t} (3 \cos 6t + \sin 6t)$$

and consequently (1) is exponentially stable.

The system presented in Example 1 is in some sense "too fast" in order that condition $\operatorname{Re} \sigma(A(t)) \leq -1$ for all $t \geq 0$ implies exponential stability. Various assumptions on the parameter variation of $A(t)$ are known, such that if $\delta > 0$ is sufficiently small then anyone of the following conditions guarantees exponential stability of (1):

$$\|\dot{A}(t)\| \leq \delta \text{ for all } t \geq 0 \quad [11] \quad (13)$$

$$\|A(t_2) - A(t_1)\| \leq \delta |t_2 - t_1| \text{ for all } t_1, t_2 \geq 0 \quad [2] \quad (14)$$

$$\sup_{0 \leq \tau \leq h} \|A(t + \tau) - A(t)\| \leq \delta \quad (15)$$

As a consequence of the following Theorem 1, (15) implies exponential stability if δ is small enough. (15) is less restrictive than a similar condition in [9], Lemma 3:

$$\limsup_{t \rightarrow \infty} \sup_{0 \leq \tau \leq h} \|A(t + \tau) - A(t)\| = 0 \text{ for all } h \geq 0 \quad (16)$$

The disadvantage of (13) - (15) is that they are qualitative conditions in the sense that δ must be small enough. We can improve the results and give quantitative bounds.

Theorem 1 Suppose $A(t) \in PC(R_+, R^{n \times n})$ satisfies for some $\alpha, M > 0$ and all $t \geq 0$

$$\|A(t)\| \leq M \quad (17)$$

$$\sigma(A(t)) \subset \{s \in C : \operatorname{Re} s < -\alpha\}.$$

Then the system (1) is exponentially stable if one of the following conditions holds true for all $t \geq 0$:

$$(i) -\alpha < -4m$$

$$(ii) A(\cdot) \text{ is piecewise differentiable and}$$

$$\|\dot{A}(t)\| \leq \delta < \frac{2}{2n-1} \frac{\alpha^{4n-2}}{2M^{4n-4}}.$$

Proof: We will use the following important inequality due to [2]:

$$\|e^{A(t)\sigma}\| \leq \left(\frac{2M}{\varepsilon} \right)^{n-1} e^{(-\alpha+\varepsilon)\sigma} \quad (18)$$

for all $\sigma, t \geq 0$ and for all $\varepsilon \in (0, 2M)$.

For fixed $t \geq 0$ (1) can be rewritten in the form

$$\dot{x}(t) = A(t_0)x(t) + (A(t) - A(t_0))x(t) \quad (19)$$

and for $x(t_0) = x_0$ its solution is given by

$$x(t) = e^{A(t_0)(t-t_0)} x_0 + \int_{t_0}^t e^{A(t_0)(t-s)} (A(s) - A(t_0)) x(s) ds \quad (20)$$

Hence by (18)

$$\begin{aligned} \|x(t)\| &\leq k_\varepsilon e^{(-\alpha+\varepsilon)(t-t_0)} \|x_0\| + \\ &+ k_\varepsilon \int_{t_0}^t e^{(-\alpha+\varepsilon)(t-s)} \|A(s) - A(t_0)\| \|x(s)\| ds \end{aligned} \quad (21)$$

for all $t \geq t_0$, where $k_\varepsilon = \left(\frac{2M}{\varepsilon}\right)^{n-1}$.

Multiplying this inequality by $e^{(\alpha-\varepsilon)t}$ and applying Gronwall's Lemna yields

$$\begin{aligned} &\|e^{(\alpha-\varepsilon)t} x(t)\| \leq \\ &\leq k_\varepsilon e^{(\alpha-\varepsilon)t_0} \|x_0\| \exp \left[k_\varepsilon \int_{t_0}^t \|A(s) - A(t_0)\| \|x(s)\| ds \right]. \end{aligned} \quad (22)$$

Thus

$$\begin{aligned} &\|x(t)\| \leq \\ &\leq k_\varepsilon e^{(-\alpha+\varepsilon)(t-t_0)} \|x_0\| \exp \left[k_\varepsilon \int_{t_0}^t \|A(s) - A(t_0)\| \|x(s)\| ds \right] \end{aligned} \quad (23)$$

Now we prove the statements (i) and (ii).

(i): Since $\|A(s) - A(t_0)\| \leq 2M$ for all $s, t_0 \geq 0$, (23) implies for $\varepsilon \in (0, 2M)$ and some $h > 0$:

$$\begin{aligned} \|x(t)\| &\leq k_\varepsilon \exp[(-\alpha + \varepsilon + 2k_\varepsilon M)(t - t_0)] \|x(t_0)\| \leq \\ &\leq k_\varepsilon \exp[(\varepsilon + 2k_\varepsilon M - 4M - h)(t - t_0)] \|x(t_0)\|. \end{aligned} \quad (24)$$

The function $f : (0, 2M] \rightarrow R$ defined as $f(\varepsilon) = \varepsilon + 2k_\varepsilon M - 4M - h$ is continuous and $f(2M) = -h$. Thus there exists $\varepsilon \in (0, 2M]$ such that $f(\varepsilon) < 0$.
(ii): Consider

$$R(t) = \int_0^\infty e^{A^T(t)s} e^{A(t)s} ds \quad (25)$$

which solves Lyapunov equation

$$R(t)A(t) + A^T(t)R(t) = -I \quad (26)$$

and satisfies for some $c_1, c_2 > 0$

$$c_1 I \leq R(t) \leq c_2 I \quad (27)$$

The derivative of $R()$ is given by

$$\dot{R}(t) = \int_0^\infty e^{A^T(t)s} [R(t)\dot{A}(t) + \dot{A}^T(t)R(t)] e^{A(t)s} ds \quad (28)$$

Now we show that

$$V(x, t) = \langle R(t)x(t), x(t) \rangle \quad (29)$$

is a Lyapunov function of (1). Its time derivative along any solution is

$$\frac{d}{dt} V(x(t), t) = \langle (-I + R(t))x(t), x(t) \rangle. \quad (30)$$

We have to show that

$$\dot{R}(t) < I \quad (31)$$

for all $t \geq 0$. Applying Coppel's inequality (18) to (28) and (25) yields

$$\|\dot{R}(t)\| \leq 2 \left(\frac{2M}{\varepsilon} \right)^{4(n-1)} \left(\frac{1}{2(-\alpha + \varepsilon)} \right)^2 \delta \quad (32)$$

and thus (31) holds if for some $\varepsilon \in (0, \alpha)$

$$\delta < 2 \left(\frac{\varepsilon}{2M} \right)^{4(n-1)} (\alpha - \varepsilon)^2. \quad (33)$$

Define a function $g : (0, \alpha) \rightarrow R$, by

$$g(\alpha) = 2 \left(\frac{\varepsilon}{2M} \right)^{4(n-1)} (\alpha - \varepsilon)^2. \text{ It is easily verified, that } g(\cdot)$$

achieves its minimum on $(0, \alpha)$ at $\varepsilon_0 = \frac{n}{n+1}\alpha$. This verifies (33) and ends the proof.

3. DISCRETE TIME SYSTEM

For system (2) we define

$$\Phi(t, t) = I \text{ and}$$

$$\Phi(t, s) = A(t-1)A(t-2)\dots A(s) \text{ for } t > s, s \in N. \quad (34)$$

We will consider system (2) under the assumptions that there exists a constant $M > 0$ such that

$$\|A(k)\| \leq M \text{ for all } k \in N \quad (35)$$

We have the following definition.

Definition 3 The system (2) is said to be exponentially stable if there exist $C, \omega > 0$, $\omega < 1$ such that

$$\|\Phi(t, s)\| \leq C\omega^{(t-s)} \text{ for all } t \geq s \geq 0. \quad (36)$$

It appears that exponential stability can be characterized in terms of Bohl exponents.

Definition 4 The Bohl exponent $\beta(A)$ of the system (2) is given by

$$\beta(A) = \limsup_{s, t \rightarrow \infty} \|\Phi(t, s)\|^{\frac{1}{t-s}}. \quad (37)$$

Moreover for the Bohl exponent we have several alternative formulas (see [10])

$$\begin{aligned} \beta(A) &= \lim_{t \rightarrow \infty} \left(\sup_{s \in N} \|\Phi(t+s, s)\|^{\frac{1}{t}} \right) = \inf_{t \in N} \left(\sup_{s \in N} \|\Phi(t+s, s)\|^{\frac{1}{t}} \right) = \\ &= \inf \{ r : \exists M_r > 0 \forall t > s > 0 \|\Phi(t, s)\| \leq M_r r^{t-s} \} \end{aligned} \quad (38)$$

and system (2) is exponentially stable if and only if $\beta(A) < 1$.

For discrete time-varying systems similarly as for continuous, exponential stability does neither imply

$$\rho(A(t)) < 1 \text{ for all } t \in N \quad (39)$$

nor does the condition

$$\rho(A(t)) < 1 \text{ for all } t \in N \quad (40)$$

guarantee exponential stability.

Denote $\delta(A) = \sup_{t \in N} \rho(A(t))$. In [5] the following theorem has been proved.

Theorem 2 For each bounded sequence $(A(t))_{t \in N}$ of matrices there exists a constant $C > 0$ such that for any $\varepsilon > 0$ we have

$$\|A^k(t)\| < C(\delta(A) + \varepsilon)^k. \quad (41)$$

In the proof of the main result of this section we will use the following discrete version of Gronwall's inequality [1].

Theorem 3 Suppose that for two sequences $(u(k))_{k \in N}$ and $(f(k))_{k \in N}$ of real numbers the following inequality

$$u(k) \leq p + q \sum_{i=0}^{k-1} u(i) f(i) \quad (42)$$

holds for certain $p, q \in R$ and all $k \in N$, then

$$u(k) \leq p \prod_{i=0}^{k-1} (1 + q f(i)) \quad (43)$$

for all $k \in N$.

For fixed $k_0 \geq 0$ (2) can be rewritten in the form

$$x(k+1) = A(k_0)x(k) + (A(k) - A(k_0))x(k) \quad (44)$$

and for $x(k_0) = x_0$ its solution is given by

$$x(k) = A^{k-k_0}(k_0)x_0 + \sum_{l=k_0}^{k-1} A^{k-l-1}(k_0)(A(l) - A(k_0))x(l). \quad (45)$$

Hence by (41)

$$\begin{aligned} \|x(k)\| &\leq C(\delta(A) + \varepsilon)^{k-k_0} \|x_0\| + \\ &+ \sum_{l=k_0}^{k-1} C(\delta(A) + \varepsilon)^{k-l-1} \|A(l) - A(k_0)\| \|x(l)\| \end{aligned} \quad (46)$$

Multiplying this inequality by $C(\delta(A) + \varepsilon)^{-k}$ yields

$$\begin{aligned} C(\delta(A) + \varepsilon)^{-k} \|x(k)\| &\leq C(\delta(A) + \varepsilon)^{-k_0} \|x_0\| + \\ &+ (\delta(A) + \varepsilon)^{-1} \sum_{l=k_0}^{k-1} C(\delta(A) + \varepsilon)^{-l} \|A(l) - A(k_0)\| \|x(l)\|. \end{aligned} \quad (47)$$

Applying Gronwall's inequality and taking into account that $\|A(k) - A(k_0)\| < M$ we obtain

$$\begin{aligned} \|x(k)\| &\leq C(\delta(A) + \varepsilon)^{k-k_0} \|x_0\| \left(1 + \frac{2M}{\rho(A) + \varepsilon} \right)^k = \\ &= C(\delta(A) + \varepsilon)^{k-k_0} \|x_0\| (\rho(A) + \varepsilon + 2M)^k. \end{aligned} \quad (48)$$

We have proved the following theorem.

Theorem 4 If $\rho(A) + 2M < 1$, then system (2) is exponentially stable.

4. NUMERICAL EXAMPLE

Consider system (2) with

$$A(k) = \begin{bmatrix} \frac{k+18}{8k+80} & \frac{1}{4} \\ 0 & \frac{k+26}{16k+160} \end{bmatrix}. \quad (49)$$

It can be show that matrix has the form

$$A(k) = \begin{bmatrix} \frac{1}{8} + \frac{1}{k+10} & \frac{1}{4} \\ 0 & \frac{1}{16} + \frac{1}{k+10} \end{bmatrix}, \text{ so we can write}$$

$$A(k) = A + \overline{A(k)}, \quad \text{where} \quad A = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} \\ 0 & \frac{1}{16} \end{bmatrix} \quad \text{and}$$

$$\overline{A(k)} = \frac{1}{k+10} I \text{ is the rate of change of } A(k).$$

Then $\sigma(A(k)) = \left\{ \frac{1}{8} + \frac{1}{k+10}, \frac{1}{16} + \frac{1}{k+10} \right\}$ thus spectral

radius is $\rho(A(k)) = \frac{1}{8} + \frac{1}{k+10}$ and chosen matrix norm

$$\|A(k)\| \leq \frac{1}{4} \text{ as a upper bounds.}$$

For matrix $A(k)$ the stability condition using **Theorem 4** is given by:

$$\begin{aligned} \sup_{k \in N} \rho(A(k)) + 2M &< 1 \\ \frac{1}{8} + \frac{1}{10} + 2 \cdot \frac{1}{4} &= \frac{58}{80} < 1. \end{aligned} \quad (50)$$

The discrete time system where we switched between defined matrices of the form $A(k)$ is exponentially stable because the stability condition holds.

5. CONCLUSION

For stationary discrete and continuous systems in the literature there are described the stability conditions. For complex systems, where state depends on switching signal, the stability conditions are not so obvious. We consider stability conditions for slowly varying systems where the parameter variation of $A(t)$, $(A(k))$ is "slow enough". In this paper not only qualitative but also quantitative stability conditions for such systems are determined. Very important is that derived conditions use only information about matrices. For check, if discrete system is exponentially stable, we need only information about eigenvalues, spectral radius and matrix norm. Important is that, the derived stability condition for discrete time system doesn't depend on order of matrices.

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